Laplace - Modified Decomposition Method for the Generalized Hirota-Satsuma Coupled KdV Equation

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Abstract
Analytical and numerical solutions are obtained for Generalized Hirota-Satsuma Coupled KdV Equation by the well-known Laplace Modified decomposition method [LMDM]. We combined Laplace transform and Modified decomposition method and present new approach for Generalized Hirota-Satsuma Coupled KdV Equation. The method does not need linearization, weak nonlinearity assumptions, or perturbation theory. We compared the numerical solutions with corresponding analytical solutions.

1. Introduction

Nonlinear coupled partial differential equations are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves and chemical physics. Several nonlinear physical phenomena can be explained by the exact and numerical solutions of nonlinear equations. In this paper, we consider a generalized Hirota-Satsuma coupled MKdV equation which was introduced by Wu et al. In [1], the authors introduced a matrix spectral problem with three potentials and proposed a corresponding hierarchy of nonlinear equations. The coupled MKdV system is very complicated and not easy to solve by direct integration method and the homogeneous balance method et al.

The generalized Hirota-Satsuma coupled KdV and coupled MKdV equations have been studied by many authors via different approaches. Recently, Fan [2] has provided a suggestion to construct soliton solutions for these equations by extended tanh-function method. Solitary solutions for various nonlinear wave equations have been investigated using different methods which can only solve special kind of nonlinear problems due to the limitations or shortcomings in the methods. The type of equations we are handing is attracting many researches and a great deal of work has already

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been done, for example, Jacobi elliptic function method by Yu et al. [3], the projective Riccati
equations method by Yong and Zhang [4], the algebraic method by Zayed et al. [5], variational
iteration method by He and Wu [6], Adomian decomposition method by Kaya [7] and homotopy
perturbation method by Ganji and Rafei [8]. In this paper; we shall consider Generalized Hirota-
Satsuma Coupled KdV Equation in the form

\[
E_t = \frac{1}{2} E_{xxx} - 3EE_x + 3(\eta \psi)_x \\
\eta_t = -\eta_{xxx} + 3E \eta_x \\
\psi_t = -\psi_{xxx} + 3E \psi_x
\] (1)

Where \( E_t = \frac{\partial E}{\partial t}, E_x = \frac{\partial E}{\partial x}, \eta_t = \frac{\partial \eta}{\partial t}, \eta_x = \frac{\partial \eta}{\partial x}, \psi_t = \frac{\partial \psi}{\partial t}, \psi_x = \frac{\partial \psi}{\partial x} \), is considered a first ,
second and third order partial differential operator In this work, we used Laplace modified
decomposition method [9, 10] which is used by Yusufoglu to solve Duffing equation [11] and
Elgasery for Falkner- Skan equations [12] , Schrödinger -KdV (Sch-KdV) equation[13], Solitary
Wave Solutions of Schrödinger Equation [14]. This decomposition method technique, extended by
Hussain and Khan [15], is a very good example of applications of Laplace transform to approximate
the solutions of the nonlinear partial differential equations [13-15].

2. Analysis of the Method

The method consists of first taking the Laplace transform of both sides of equations in system (1)
with differentiation property of Laplace transform and taking inverse Laplace transform. We get

\[
E(x,t) = E(x,0) + L^{-1} \left[ \frac{1}{p} L \left( \frac{1}{2} E_{xxx} - N + M \right) \right]
\]

\[
\eta(x,t) = \eta(x,0) + L^{-1} \left[ \frac{1}{p} L(R - \eta_{xxx}) \right]
\]

\[
\psi(x,t) = \psi(x,0) + L^{-1} \left[ \frac{1}{p} L(Q - \psi_{xxx}) \right]
\] (2)

Where \( N(E,\eta,\psi) = 3EE_x, M(E,\eta,\psi) = 3(\eta \psi), R(E,\eta,\psi) = 3E \eta_x, Q(E,\eta,\psi) = 3E \psi_x \) are symbolize nonlinear
unknown function \( E(x, t) \) \( \eta(x, t) \) and \( \psi(x, t) \) by the infinite series given by

\[
\begin{align*}
E(x,t) &= \sum_{n=0}^{\infty} E_n \\
\eta(x,t) &= \sum_{n=0}^{\infty} \eta_n \\
\psi(x,t) &= \sum_{n=0}^{\infty} \psi_n
\end{align*}
\] (3)
The nonlinear terms \( N(E, \eta, \psi) = 3EE \), \( M(E, \eta, \psi) = 3(\eta \psi) \), \( R(E, \eta, \psi) = 3E\eta \), \( Q(E, \eta, \psi) = 3E\psi \), are usually represented by an infinite series of the so-called Adomian polynomials [18]

\[
N(E, \eta, \psi) = \sum_{n=0}^{\infty} A_n \left[ \sum_{i=0}^{n} \phi_i E_i \right], \quad M(E, \eta, \psi) = \sum_{n=0}^{\infty} C_n \left[ \sum_{i=0}^{n} \phi_i \eta_i \right], \quad R(E, \eta, \psi) = \sum_{n=0}^{\infty} D_n \left[ \sum_{i=0}^{n} \phi_i \psi_i \right]
\]

The Adomian polynomials can be generated for all forms of nonlinearity. They are determined by the following relations:

\[
A_n = \frac{1}{n!} \left[ \frac{d^n}{d\xi^n} \left( N \left( \sum_{i=0}^{\infty} \phi_i E_i, \sum_{i=0}^{\infty} \phi_i \eta_i, \sum_{i=0}^{\infty} \phi_i \psi_i \right) \right) \right]_{\xi=0}, \quad n \geq 0
\]

Similarly

\[
B_n = \frac{1}{n!} \left[ \frac{d^n}{d\xi^n} \left( M \left( \sum_{i=0}^{\infty} \phi_i E_i, \sum_{i=0}^{\infty} \phi_i \eta_i, \sum_{i=0}^{\infty} \phi_i \psi_i \right) \right) \right]_{\xi=0}, \quad n \geq 0
\]

\[
C_n = \frac{1}{n!} \left[ \frac{d^n}{d\xi^n} \left( R \left( \sum_{i=0}^{\infty} \phi_i E_i, \sum_{i=0}^{\infty} \phi_i \eta_i, \sum_{i=0}^{\infty} \phi_i \psi_i \right) \right) \right]_{\xi=0}, \quad n \geq 0
\]

\[
D_n = \frac{1}{n!} \left[ \frac{d^n}{d\xi^n} \left( Q \left( \sum_{i=0}^{\infty} \phi_i E_i, \sum_{i=0}^{\infty} \phi_i \eta_i, \sum_{i=0}^{\infty} \phi_i \psi_i \right) \right) \right]_{\xi=0}, \quad n \geq 0
\]

This formula is easy to set computer code to get as many polynomials as we need in calculation of the numerical as well as explicit solutions. For the sake of convenience of the readers, we can give the first few Adomian polynomials for \( N(E, \eta, \psi) = 3EE \), \( M(E, \eta, \psi) = 3(\eta \psi) \), \( R(E, \eta, \psi) = 3E\eta \), \( Q(E, \eta, \psi) = 3E\psi \) of the nonlinearity as

\[
A_0 = 3E_0 E_0, \quad A_1 = 3E_0 E_{1x} + 3E_1 E_0 + 3E_1 E_1, \quad A_2 = 3E_0 E_{2x} + 3E_0 E_{2x} + 3E_1 E_{2x} + 3E_1 E_{2x} + 3E_2 E_{1x} + 3E_2 E_2, \\
C_0 = 3E_0 \eta_0, \quad C_1 = 3E_0 \eta_{1x} + 3E_1 \eta_0 + 3E_1 \eta_1, \quad C_2 = 3E_0 \eta_{2x} + 3E_0 \eta_{2x} + 3E_1 \eta_{1x} + 3E_1 \eta_{1x} + 3E_2 \eta_{1x} + 3E_2 \eta_{1x} + 3E_2 \eta_{2x} + 3E_2 \eta_{2x}, \\
B_0 = 3(\eta_0 \psi_0)_x, \quad B_1 = 3(\eta_0 \psi_{1x})_x + 3(\eta_{1x} \psi_0)_x, \quad B_2 = 3(\eta_0 \psi_{2x})_x + 3(\eta_{1x} \psi_{1x})_x + 3(\eta_{2x} \psi_0)_x + 3(\eta_{2x} \psi_{1x})_x + 3(\eta_{2x} \psi_{2x})_x \\
D_0 = 3E_0 \psi_0, \quad D_1 = 3E_0 \psi_{1x} + 3E_1 \psi_0 + 3E_1 \psi_{1x}, \quad D_2 = 3E_0 \psi_{2x} + 3E_0 \psi_{2x} + 3E_1 \psi_{2x} + 3E_1 \psi_{2x} + 3E_2 \psi_{2x} + 3E_2 \psi_{2x} + 3E_2 \psi_{2x} + 3E_2 \psi_{2x}
\]

and so on, the rest of the polynomials can be constructed in a similar manner. Substituting the initial conditions into (2) identifying the zeroth components \( E_0 \) and \( \eta_0 \) then we obtain the subsequent components by the following recursive equations by using the standard ADM

\[
E_{n+1} = L^{-1} \left[ \frac{1}{p} L \left( \frac{1}{2} E_{nxx} - A_n + B_n \right) \right], \quad n \geq 0
\]

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\[ \eta_{n+1} = L^{-1} \left[ \frac{1}{p} L \left( C_n - \eta_{n_{\text{ass}}} \right) \right], n \geq 0 \tag{5} \]
\[ \psi_{n+1} = L^{-1} \left[ \frac{1}{p} L \left( D_n - \psi_{n_{\text{ass}}} \right) \right], n \geq 0 \]

Recently, Wazwaz [19] proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. In [19], he assumed that if the zeroth component \( E_0 = g \) and the function \( g \) is possible to divide into two parts such as \( g_1 \) and \( g_2 \), the one can formulate the recursive algorithm for \( E_0 \) and general term \( E_{n+1} \) in a form of the modified recursive scheme as follows:

\[ E_0 = g_1, \]
\[ E_1 = g_2 + L^{-1} \left[ \frac{1}{p} L \left( \frac{1}{2} E_{n_{\text{ass}}} - A_0 + B_0 \right) \right], \]
\[ E_{n+1} = L^{-1} \left[ \frac{1}{p} L \left( \frac{1}{2} E_{n_{\text{ass}}} - A_n + B_n \right) \right], n \geq 1 \tag{6} \]

Similarly, if the zeroth component \( \eta_0 = h \) and the function \( h \) is possible to divide into two parts such as \( h_1 \) and \( h_2 \)

\[ \eta_0 = h_1 \]
\[ \eta_1 = h_2 + L^{-1} \left[ \frac{1}{p} L \left( C_0 - \eta_{0_{\text{ass}}} \right) \right], \]
\[ \eta_{n+1} = L^{-1} \left[ \frac{1}{p} L \left( C_n - \eta_{n_{\text{ass}}} \right) \right], n \geq 1 \tag{7} \]

if the zeroth component \( \psi_0 = \lambda \) and the function \( \lambda \) is possible to divide into two parts such as \( \lambda_1 \) and \( \lambda_2 \)

\[ \psi_0 = \lambda_1 \]
\[ \psi_1 = \lambda_2 + L^{-1} \left[ \frac{1}{p} L \left( D_0 - \psi_{0_{\text{ass}}} \right) \right], \]
\[ \psi_{n+1} = L^{-1} \left[ \frac{1}{p} L \left( D_n - \psi_{n_{\text{ass}}} \right) \right], n \geq 1 \tag{8} \]

Complicated nonlinear differential equations can also be solved by applying ADM after this type of modifications. It is worth noting that the zeros components \( E_0, \eta_0 \) and \( \psi_0 \) are defined then the remaining component \( E_n \) and \( \eta_n, \eta_{n_{\text{ass}}} \), \( n \geq 1 \) can be completely determined. As a result, the components \( E_0, E_1, \ldots, \eta_0, \eta_1 \ldots \) and \( \psi_0, \psi_1 \ldots \) are estimated and the series solutions can be obtained and in many cases the exact solution in a closed form can also be established.

The decomposition series (3) solutions are generally converges very rapidly in real physical problems [18]. The rapidity of this convergence means that few terms are required. Convergence of this method has been rigorously established by Cherruault [20], Abbaoui and Cherruault [21,22]
and Himoun, Abbaoui and Cherruault [2]. The practical solutions will be the \( n \)-term approximations \( \chi_n, \mu_n \) and \( \tau_n \)

\[
\chi_n = \sum_{i=0}^{n-1} E_i(x,t), \quad n \geq 1 \\
\mu_n = \sum_{i=0}^{n-1} \eta_i(x,t), \quad n \geq 1 \\
\tau_n = \sum_{i=0}^{n-1} \psi_i(x,t), \quad n \geq 1
\]

\[
\lim_{n \to \infty} \chi_n = E(x,t) \\
\lim_{n \to \infty} \mu_n = \eta(x,t) \\
\lim_{n \to \infty} \tau_n = \psi(x,t)
\]

(9)

2.1. Implementation of the method

We consider the application of Hirota-Satsuma Coupled KdV Equations with the exact solution are given in [25] as

\[
E(x,t) = \frac{1}{3}(\alpha - 2k^2) + 2k^2 \tanh^2[k(x + \alpha t)]
\]

\[
\eta(x,t) = -4k^2z_0(\alpha + k^2) - 4k^2(\alpha + k^2) \tanh[k(x + \alpha t)]
\]

\[
\psi(x,t) = z_0 + z_1 \tanh[k(x + \alpha t)]
\]

(10)

To solve the system of Eq. (1) by means of Laplace-modified decomposition method, and for simplicity we take, \( \{k = \alpha = z_0 = z_1 = 1\} \) we get initial condition

\[
E(x,0) = -\frac{1}{3} + 2 \tanh^2(x), \quad \eta(x,0) = -\frac{8}{3} + \frac{8}{3} \tanh(x), \quad \psi(x,0) = 1 + \tanh(x)
\]

(11)

Using (5), (6), (7) and (8) with (4), (11) and we get

\[
E_0 = 0
\]

\[
E_1 = E(x,0) + L^{-1}\left[\frac{1}{p} L\left[\frac{1}{2} E \left( 0_{xxx} - A_0 + B_0 \right) \right]\right] = -\frac{1}{3} + 2 \tanh^2(x)
\]

\[
E_2 = L^{-1}\left[\frac{1}{p} L\left(\frac{1}{2} E_{xxx} - A_1 + B_1 \right) \right] = 4t \sec h^2(x) \tanh(x)
\]

\[
E_3 = L^{-1}\left[\frac{1}{p} L\left(\frac{1}{2} E_{2xxx} - A_2 + B_2 \right) \right]
\]

\[
= t^2(- \sec h^6(x) + 6 \sec h^2(x) \tanh^4(x) + 10 \sec h^4(x) - 10 \sec h^2(x) \tanh^2(x))
\]

\[
- \frac{16}{3} t^3 \sec h^4(x) \tanh(x) \left[3 \sec h^6(x) - 6 \tanh^2(x) - 2 \right]
\]

\[
\eta_0 = 0
\]

\[
\eta_1 = \eta(x,0) + L^{-1}\left[\frac{1}{p} L(C_0 - \eta_0_{xxx}) \right] = -\frac{8}{3} + \frac{8}{3} \tanh(x)
\]

\[
\eta_2 = L^{-1}\left[\frac{1}{p} L(C_1 - \eta_1_{xxx}) \right] = \frac{8t}{3} \sec h^4(x)
\]
\[ \eta_3 = L^{-1} \left[ \frac{1}{p} L(C_2 - \eta_{2xx}) \right] = -\frac{8}{3} t^2 \sec h^2(x) \tanh(x) + \frac{64}{3} t^3 \sec h^4(x) \tanh^2(x) \]

\[ \psi_0 = 0 \]

\[ \psi_1 = \psi(x,0) + L^{-1} \left[ \frac{1}{p} L(D_0 - \psi_{0xx}) \right] = 1 + \tanh(x) \]

\[ \psi_2 = L^{-1} \left[ \frac{1}{p} L(D_1 - \psi_{1xx}) \right] = t \sec h^2(x) \]

\[ \psi_3 = L^{-1} \left[ \frac{1}{p} L(D_2 - \psi_{2xx}) \right] = t^2 \sec h^2(x) \tanh(x) \left[ \sec h^2(x) + 2 \right] - 8t^3 \sec h^4(x) \tanh^2(x) \]

Similarly, we can also find other components and the approximate solution for calculating \( \eta \)th terms as follows:

\[ E(x,t) = -\frac{1}{3} t^2 \tanh^2(x) + 4t \sec h^2(x) \tanh(x) \]

\[ + t^2 \left( -\sec h^6(x) + 6 \sec h^2(x) \tanh^4(x) + 10 \sec h^4(x) - 10 \sec h^2(x) \tanh^2(x) \right) \]

\[ - \frac{16}{3} t^3 \sec h^4(x) \tanh(x) \left[ 3 \sec h^6(x) - 6 \tanh^2(x) - 2 \right] \]

\[ \eta(x,t) = \frac{-8}{3} + \frac{8}{3} \tanh(x) + \frac{8}{3} t \sec h^2(x) + -\frac{8}{3} t^2 \sec h^2(x) \tanh(x) + \frac{64}{3} t^3 \sec h^4(x) \tanh^2(x) \]

and

\[ \psi(x,t) = 1 + \tanh(x) + t \sec h^2(x) + t^2 \sec h^2(x) \tanh(x) \left[ \sec h^2(x) + 2 \right] - 8t^3 \sec h^4(x) \tanh^2(x) \]

(12)

**Figure 1.** Exact and numerical solution of \( E(x,t) \), \(-10 \leq x \leq 10, -1 \leq t \leq 1 \).

**Figure 2.** Exact and numerical solution of \( \eta(x,t) \), \(-10 \leq x \leq 10, -1 \leq t \leq 1 \).
The component wise exact and numerical solution of system (1) with 7th terms by (LMDM) are shown from Figure. 1 to 3.

3. Conclusion

The Laplace Modified decomposition method [LMDM] is a powerful method which has provided an efficient potential for Generalized Hirota-Satsuma Coupled KdV Equation with initial condition. The approximate solutions to the equations have been calculated by using the method [LMDM] without any need to a transformation techniques and linearization of the equations. Additionally, it does not need any discretization method to get numerical solutions. This method thus eliminates the difficulties and massive computation work. The algorithm can be used without any need to complex calculations except for simple and elementary operations.

References


