Canadian Journal of Basic and Applied Sciences

ISSN 2292-3381

CJBAS Vol. 02(03), 64-75, 2014

A Mixed Finite Element for Evaluation of Interlaminar Stresses in Composite Structures

R. Marimuthu [a], *B. Nageswara Rao* [b*]

[a] Structural Analysis and Testing Group, Vikram Sarabhai Space Centre, Trivandrum-695 022, India
[b] Department of Mechanical Engineering, School of Civil and Mechanical Sciences, KL University, Green Fields, Vaddeswaram – 522 502, India

Keywords:

Composite Structures, Interlaminar stresses, Mixed finite element, Transverse, displacements

Abstract

A 20-node brick element having three translations and three interlaminar stresses as degrees of freedom per node is considered for evaluation of interlaminar stresses in composite structures. In the mixed finite element formulation, virtual energy due to interlaminar stresses and strains are added to the standard virtual strain energy function and minimized with respect to nodal displacements and interlaminar stresses. Element stiffness matrices are found to be unsymmetric. Frontal solution program for unsymmetric matrices is used to obtain the solution. Finite element analysis results on laminated composite plates under transverse loads are matching well with existing solutions.

1. Introduction

The finite element method has the capability to deal with complex loading conditions, material behavior and practical geometries. The finite element methods for analyzing the stress and displacement distributions of an elastic continuum have long been interpreted as approximate methods associated with different variational principles in elasticity. These displacement and/or stress fields are then assumed in each element and the resulting equations from the application of the variational principles are simultaneous algebraic equations, which may have: generalized displacements; generalized internal forces or stresses; or both displacements and forces at the

nodal points as unknowns to be evaluated. Based on the nature of the final matrix equations, the above three categories of finite element methods are often referred to as: displacement method; force method; and mixed method.

Laminated composites occupy a major portion of the advanced launch vehicle structures. Plate or shell theories are being used to examine the overall deformation of thin laminated structures.

* Corresponding Author:
E-mail, *bnrao52@rediffmail.com* - Tel, (+91) 86456948 - Fax, (+91) 8645247249

Each lamina in a laminate is assumed to be perfectly bonded with its adjacent lamina, which implies continuity of normal and shear stresses in thickness direction (interlaminar stresses). These stresses are useful for assessing the load bearing capacity of composite structures under complex service conditions. Finite elements of the displacement formulation generally provide accurate displacements and in-plane stresses, whereas normal and shear stresses may not be continuous in the laminate thickness direction.

Delamination due to the presence of interlaminar stresses may cause reduction in the load bearing capacity of the composite structures. Pipes and Pagano [1] and Bhaskar et al. [2] have utilized the finite difference method to evaluate interlaminar stresses in composite laminates. Wang and Crossman [3] have carried out finite element analysis to study edge effect in symmetric composite laminates. Wang and Choi [4] have studied boundary layer effects in composites. Wei and Zhao [5] have carried out three-dimensional finite element analysis for evaluating interlaminar stresses of symmetric laminates. Finite element analysis has been carried out for obtaining interlaminar stresses in a variety of laminate configurations [6-11]. These finite element analysis results are obtained from the displacement formulation. Stresses evaluated from the derivatives of displacements and the material constants may not guarantee the continuity of stresses at the element boundaries. Liu and Jou [12] have introduced interlaminar stresses as additional degrees of freedom and solved the generalized plane strain problem. Chaudhuri’s [13] formulation is limited to triangular elements in which the transverse shear stresses evaluated in thick composite plates. Pagano [14] has provided exact solutions for composite laminates in cylindrical bending useful for validation of the numerical solutions. Advanced formulations allow the description of the bending behavior of thin structures in an accurate way [15-17]. Each layer is discretized with several elements (≈ 5 to 10) in thickness direction, which demands more computing time.

Classical laminated plate theory (CPT) has been extensively used in the design and analysis of structural laminates. However, the first-order shear deformation theory (FSDT) gives satisfactory results [18, 19]. Reddy [20] provided a layer-wise theory by representing separately the displacement field in each layer. Zig-zag theories were developed for multilayered plates and shells by evolving the piecewise polynomial distributions of the membrane displacements in thickness direction by introducing additional variables for each layer [21]. Many finite elements have been proposed based on the higher-order shear deformation theory (HSDT) models [22-24]. Gruttmann and Wagner [25] and Schung et al. [26] have suggested a procedure to evaluate the interlaminar shear stresses in layered composite plates. Motivated by the work of the above researchers, this paper examines the adequacy of interlaminar stresses evaluated in a laminated plate under transverse load utilizing the developed 20-node layered hexahedron element based on the mixed finite element formulation.

2. Formulation

Figure 1 shows a composite laminated plate having ‘nl’ layers of orthotropic lamina. Each lamina is assumed to be perfectly bonded with its adjacent lamina, which implies continuity of normal stresses $(\sigma_x, \sigma_y, \sigma_z)$ and shear stresses $(\tau_{xy}, \tau_{yz}, \tau_{zx})$ in thickness direction (interlaminar stresses). Marimuthu et al. [16, 17] have developed a mixed finite element by imposing the interlaminar stresses as constraint to the regular finite element. A 20-node layered hexahedron element is utilized considering three displacements (u, v, w) and the three interlaminar stresses $(\sigma_z, \tau_{xz}, \tau_{yz})$ as the nodal degrees of freedom.

Applying the principle of virtual work, one can write

$$\sum_{k=1}^{nl} \int_{V_k} (\delta\varepsilon^T \sigma + \delta\sigma_2^T (\varepsilon_2 - \bar{S}\sigma)) dV_k = 0 \tag{1}$$

Where k varies from 1 to number of layers (nl) within each element; The constitutive relation: $\varepsilon = S\sigma$, in which $\varepsilon = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xz}, \gamma_{yz}, \gamma_{xy}\}^T$, is the strain, S is the transformed compliance, and $\sigma = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xz}, \tau_{yz}, \tau_{xy}\}^T$, is the stress. The strain, $\varepsilon_2 = \{\varepsilon_z, \gamma_{xz}, \gamma_{yz}\}^T$ is related to the stress (σ) by the reduced compliance $(\bar{S})$ as: $\varepsilon_2 = \bar{S}\sigma = S_1\sigma_1 + S_2\sigma_2$, in which the in-plane stresses are grouped by $\sigma_1 = \{\sigma_x, \sigma_y, \tau_{xy}\}^T$ and the interlaminar stresses are grouped by $\sigma_2 = \{\sigma_z, \tau_{xz}, \tau_{yz}\}^T$. The symbol, δ represents the virtual variation, whereas V_k is the volume of the kth layer.

The geometry of the element in Figure 1 is represented by:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{20} N_i(\xi, \eta, \varsigma) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \tag{2}$$

where Ni (i = 1 to 20) are the shape functions of the standard 20-node hexahedron element. The elemental vector, $q = \{u, v, w, \sigma_z, \tau_{xz}, \tau_{yz}\}^T$ having three displacements (viz., u, v and w) and three interlaminar stresses (viz., σ_z, τ_{xz} and τ_{yz}) as degrees of freedom at nodes.

At any point in the element, these are

$$\{q\}^e = \sum_{i=1}^{20} N_i(\xi, \eta, \varsigma)\{q_i\} \tag{3}$$

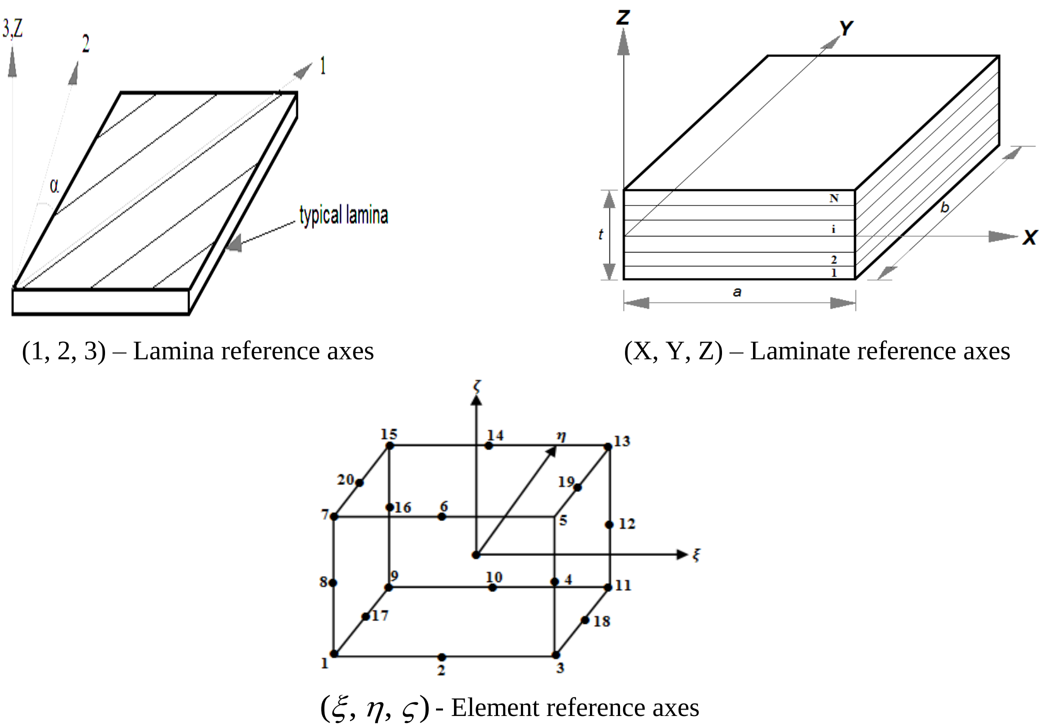

Figure 1. A composite laminated plate with N-layers of orthotropic lamina

The element strains are

$$\varepsilon = \left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right\}^T = Bq \tag{4}$$

Here B is the strain-shape function matrix.

The stress-strain relationship is written in the form

$$\sigma = D\varepsilon = DBq \tag{5}$$

The element interlaminar stresses and strains are

$$\sigma_2 = \{\sigma_z, \tau_{xz}, \tau_{yz}\}^T = N_\sigma q \tag{6}$$

$$\varepsilon_2 = \{\varepsilon_z, \gamma_{xz}, \gamma_{yz}\}^T = B_2 q \tag{7}$$

The term $\bar{S}\sigma$ in equation (1) can be expressed in the form

$$\bar{S}\sigma = S_1\sigma_1 + S_2\sigma_2 = S_1\bar{D}Bq + S_2 N_\sigma q \tag{8}$$

Here $\bar{D}$ is the reduced material matrix of the in-plane stresses.

A part of the B, N_σ and B_2 matrices applicable to the ith node are given by:

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 \end{bmatrix}; \; N_{\sigma i} = \begin{bmatrix} 0 & 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & 0 & N_i \end{bmatrix}; \text{ and } B_{2i} = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial z} & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 \end{bmatrix}.$$

Using the above relations, equation (4) can be written in the form

$$\delta q^T \left\{ \sum_{k=1}^{nl} \int_{V_k} (B^T D B + N_\sigma^T (B_2 - S_1 \overline{D} B - S_2 N_\sigma))\, dV_k \right\} q = 0 \tag{9}$$

The stiffness matrix (*K*) and the force vector (*F*) corresponding to the mixed finite element formulation is given by

$$Kq = F \tag{10}$$

The stiffness matrix (*K*) in equation (10) is

$$K = \sum_{k=1}^{nl} \int_{V_k} (B^T D B + N_\sigma^T (B_2 - S_1 \overline{D} B - S_2 N_\sigma))\, dV_k \tag{11}$$

The force vector (*F*) is given by

$$F^T = \{F_{x1}, F_{y1}, F_{z1}, 0,0,0, \ldots\ldots.., F_{x20}, F_{20y}, F_{20z}, 0,0,0\} \tag{12}$$

The stiffness matrix (K) turns out to be unsymmetrical matrix. For the assembly of element matrices and solutions, unsymmetrical frontal solver is used [27] in addition to the utilization of standard routines of the in-house developed FEAST (Finite Element Analysis of Structures) software package. It is customary to assess the developed finite elements with problems of known solutions prior to actual use. The adequacy of the present mixed finite element is examined by modeling a symmetric cross-ply laminate under uniform axial loading. The analysis results are matching well with those of Liu and Jou [12] and Marimuthu et al. [17].

3. Numerical Results

Finite element analysis has been carried out on a cross-ply laminate under cylindrical bending considering one element for each layer to have comparison with exact solution [14].

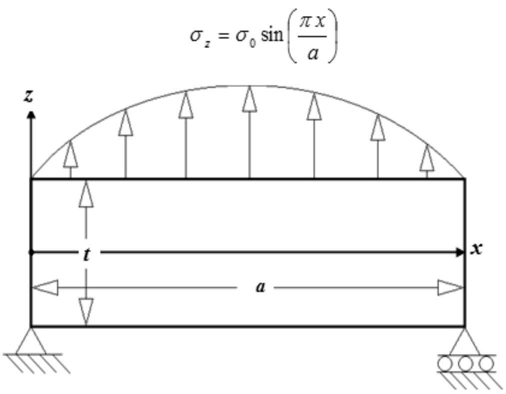

Figure 2. Schematic respresentation of a laminated plate

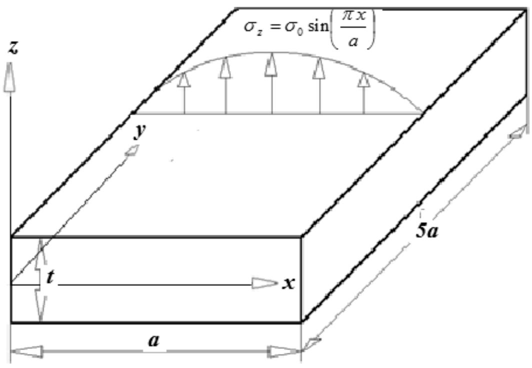

Figure 3. Plate model for the anaysis

A laminated plate (see Figure 2) is analyzed for sinusoidal load. To simulate plane strain condition, width of the plate is assumed to have five times the length as in Figure 3. Orthotropic material properties (viz., Young's modulus, shear modulus and Poisson's ratio) of a lamina specified are: EL = 172.4 GPa; ET = EZ = 6.894 GPa; GLT = GLZ = 3.477 GPa; GTZ = 1.38 GPa; and νLT = νLZ = νTZ = 0.25. Here subscript L denotes the properties in the direction parallel to the fibres, whereas subscript T is in the transverse direction, and Z is in the direction perpendicular to the LT plane.

The specified boundary conditions at edges or over surfaces are

$$u = v = w = 0 \quad \text{at x =0, z=0} \tag{13}$$

$$v = w = 0 \quad \text{at x = } a\text{, z = 0} \tag{14}$$

$$w = 0 \quad \text{at faces x = 0 and x = } a \tag{15}$$

$$\sigma_z = \sigma_0 \sin\left(\frac{\pi x}{a}\right), \; \tau_{xz} = \tau_{yz} = 0 \quad \text{Over top face, } z = \frac{t}{2} \tag{16}$$

$$\sigma_z = \tau_{xz} = \tau_{xy} = 0 \quad \text{Over bottom face, } z = -\frac{t}{2} \tag{17}$$

hexpressed as:

$$\bar{\sigma}_z = \frac{\sigma_z\left(\frac{a}{2}, z\right)}{\sigma_0}, \quad \bar{\tau}_{xz} = \frac{\tau_{xz}(0, z)}{\sigma_0}, \quad \bar{w} = \frac{100 E_T t^3}{\sigma_0 a^4} w\left(\frac{a}{2}, 0\right) \tag{18}$$

Table 1 provides the normalized displacement $(\overline{w})$ for the three geometrical configurations (viz., case-1, case-2 and case-3).

Table 1. Normalized displacement $(\overline{w})$ for different aspect ratio

Aspect ratio (S)	Case-1	Case-2	Case-3
4	1.90	4.62	2.84
10	0.73	2.94	0.93
20	0.55	2.67	0.61
30	0.51	2.60	0.55
40	0.50	2.56	0.53
50	0.50	2.54	0.52

Figures 4 to 6 show the normalized displacement plots with aspect ratio. Tables 2 and 3 provide the normalized shear stresses and normal stresses for the three geometrical configurations (case-1, case-2 and case-3). Figure-7 shows the comparison of normalized normal stress through thickness of the plate for aspect ratio S=4 of case-2. Figure 8 shows the comparison of normalized normal stress through thickness of the plate for aspect ratio, S=10 of case-3. Figures 9 to 11 show the comparison of normalized shear stress through thickness of the plate for aspect ratio, S=4 of case-1, case-2 and case-3 respectively. Figure 12 shows the normalized shear stress through thickness of the plate for S=10 of case-3.

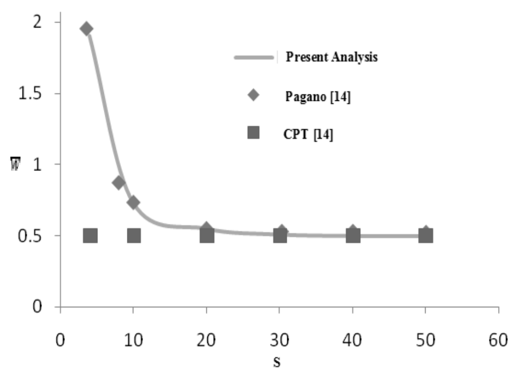

Figure 4. Normalised displacement $(\overline{w})$ with aspect ratio (Case-1)

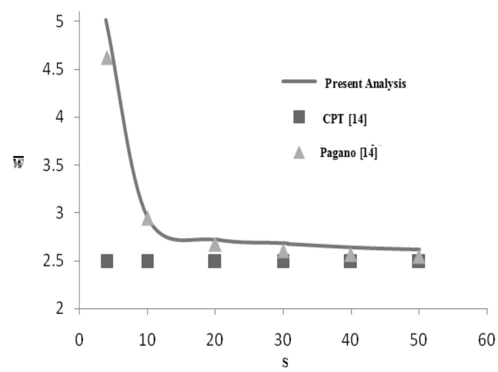

Figure 5. Normalised displacement $(\overline{w})$ with aspect ratio (Case-2)

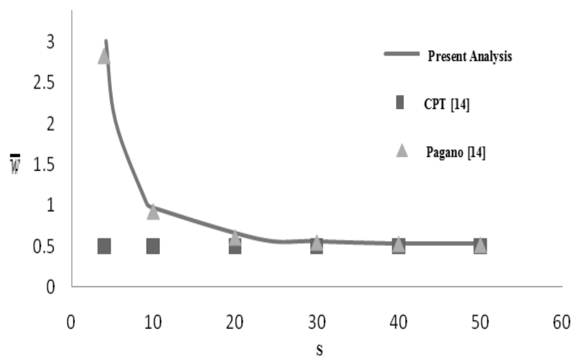

Figure 6. Normalised displacement $(\overline{w})$ with aspect ratio (Case-3)

Table 2. Normalised shear stress ($\bar{\tau}_{xz}$) for the three geometrical configurations under sinusoidal loading

$\bar{z}=\frac{z}{t}$	S=4 Case-1	S=4 Case-2	S=4 Case-3	S=10 Case-1	S=10 Case-2	S=10 Case-3
-0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.4583	0.4365	0.9606	0.5795	1.0120	2.9250	1.1620
-0.4167	0.8222	1.7980	1.0830	1.9710	5.6410	2.2580
-0.375	1.0390	2.2060	1.3200	2.5970	6.9700	2.9210
-0.3333	1.2510	2.6030	1.5490	3.2150	8.2670	3.5740
-0.2917	1.3880	2.7690	1.6370	3.6580	8.7780	3.9720
-0.25	1.5270	2.9400	1.7220	4.1010	9.2840	4.3630
-0.2083	1.6080	2.8880	1.6390	4.4060	9.1090	4.4850
-0.1667	1.6920	2.8420	1.5280	4.7100	8.9320	4.5750
-0.125	1.7370	2.5740	1.5020	4.8890	7.9980	4.5360
-0.0833	1.7840	2.3020	1.4880	5.0670	7.0460	4.5110
-0.0417	1.8010	1.7070	1.4850	5.1260	5.1090	4.5140
0.0	1.8200	1.0390	1.4830	5.1850	3.0350	4.5210
0.0417	1.8110	0.9638	1.4780	5.1270	2.8010	4.5110
0.0833	1.8050	0.9139	1.4760	5.0700	2.6190	4.5050
0.125	1.7670	0.8586	1.4830	4.8920	2.4750	4.5270
0.1667	1.7320	0.8074	1.5030	4.7150	2.3400	4.5630
0.2083	1.6550	0.7395	1.6350	4.4110	2.1410	4.4770
0.25	1.5800	0.6721	1.7380	4.1080	1.9430	4.3580
0.2917	1.4440	0.5902	1.6630	3.6650	1.7030	3.9690
0.3333	1.3090	0.5080	1.5860	3.2220	1.4620	3.5730
0.375	1.0920	0.4042	1.3580	2.6030	1.1490	2.9220
0.4167	0.8690	0.2992	1.1210	1.9770	0.8342	2.2600
0.4583	0.4623	0.1525	0.6007	1.0140	0.4216	1.1630
0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

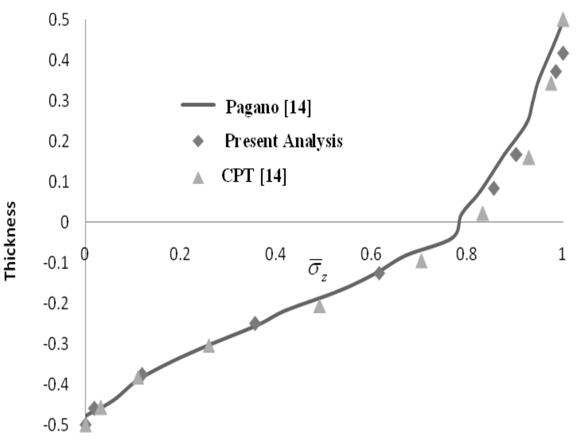

Figure 7. Normalised normal stress ($\bar{\sigma}_z$) across the thickness (t) for S=4 (Case-2)

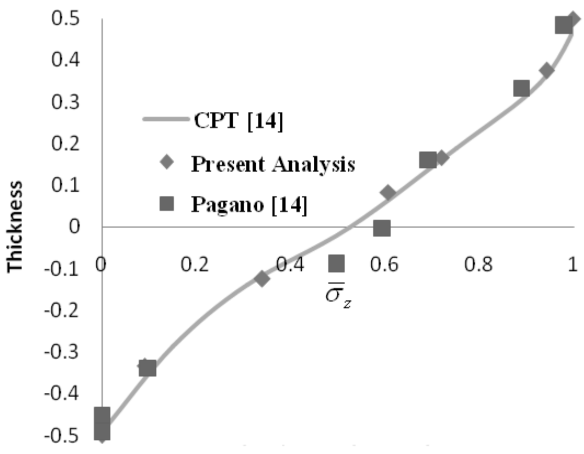

Figure 8. Normalised normal stress ($\bar{\sigma}_z$) across the thickness (t) for S=10 (Case-3)

Table 3. Normalized normal stress ($\overline{\sigma}_z$) for the three geometrical configurations under sinusoidal loading

$\bar{z}=\frac{z}{t}$	S=4			S=10		
	Case-1	Case-2	Case-3	Case-1	Case-2	Case-3
-0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.4583	0.0084	0.0193	0.0111	0.0103	0.0308	0.0106
-0.4167	0.0209	0.0486	0.0284	0.0283	0.0890	0.0307
-0.375	0.0538	0.1184	0.0691	0.0586	0.1584	0.0616
-0.3333	0.0829	0.1817	0.1061	0.0862	0.2266	0.0901
-0.2917	0.1277	0.2702	0.1576	0.1275	0.3043	0.1352
-0.25	0.1690	0.3549	0.2067	0.1653	0.3799	0.1768
-0.2083	0.2208	0.4453	0.2585	0.2171	0.4608	0.2312
-0.1667	0.2703	0.5369	0.3084	0.2660	0.5423	0.2832
-0.125	0.3257	0.6148	0.3562	0.3238	0.6191	0.3401
-0.0833	0.3799	0.6985	0.4047	0.3800	0.7008	0.3960
-0.0417	0.4370	0.7479	0.4519	0.4405	0.7458	0.4487
0.0	0.4938	0.7992	0.4993	0.5006	0.7927	0.5012
0.0417	0.5508	0.8223	0.5462	0.5607	0.8111	0.5537
0.0833	0.6083	0.8544	0.5933	0.6213	0.8403	0.6063
0.125	0.6634	0.8764	0.6410	0.6774	0.8511	0.6621
0.1667	0.7199	0.9024	0.6882	0.7353	0.8653	0.7188
0.2083	0.7709	0.9216	0.7378	0.7844	0.8817	0.7709
0.25	0.8245	0.9446	0.7898	0.8366	0.9009	0.8256
0.2917	0.8672	0.9574	0.8394	0.8737	0.9151	0.8663
0.3333	0.9141	0.9741	0.8919	0.9148	0.9308	0.9110
0.375	0.9447	0.9846	0.9302	0.9443	0.9617	0.9416
0.4167	0.9815	0.9995	0.9748	0.9790	0.9954	0.9776
0.4583	0.9911	0.9976	0.9877	0.9856	0.9907	0.9841
0.5	1.0070	1.0020	1.0080	0.9991	0.9953	0.9980

4. Concluding Remarks

A 20-node layered hexahedron element having three displacements and three interlaminar stresses as degrees of freedom is developed based on the mixed finite element formulation useful for evaluation of the interlaminar stresses at interlayer of a lamina as well as near the free edge. To examine the adequacy of the developed element, finite element analysis has been carried out on symmetric cross-ply laminates subjected to transverse sinusoidal loading. The present analysis results match well with the existing solution. The CPT solution of Pigano [14] converges to the exact solution with increasing span-to-depth ratio (S). This may be the reason why Pagano [14] in his concluding remarks stated that CPT seems to be adequate in the elastic design of very thin (high ‘S’) bodies like composite skins in aircraft structures.

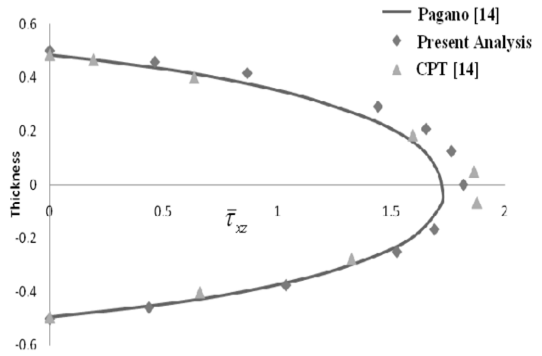

Figure 9. Normalised Shear stress ($\bar{\tau}_{xz}$) across the thickness (t) for S=4 (Case-1)

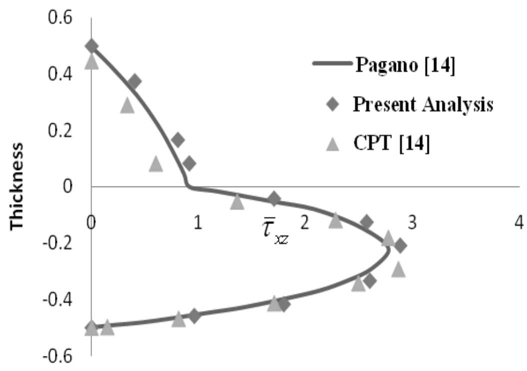

Figure 10. Normalised Shear stress ($\bar{\tau}_{xz}$) across the thickness for S=4 (Case-2)

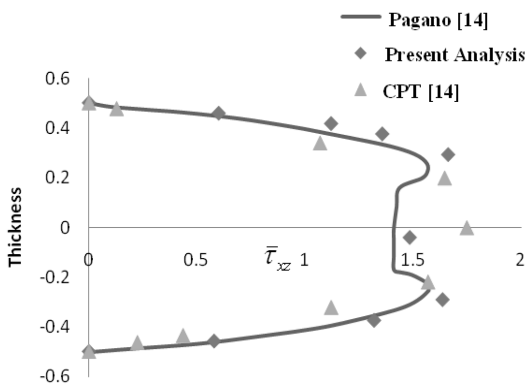

Figure 11. Normalised Shear stress ($\bar{\tau}_{xz}$) across the thickness (t) for S=4 (Case-3)

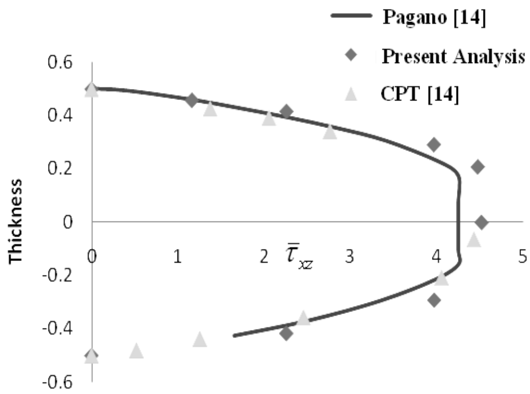

Figure 12. Normalised shear stress ($\bar{\tau}_{xz}$) across the thickness (t) for S=10 (Case-3)

It should be noted that the present analysis has been carried out considering one element for each layer. The developed element has the capability to model multiple layers for each element. Formulation is general in nature can be used for any loading and boundary conditions. The element stiffness matrix is unsymmetrical and Frontal solver developed by Hood [27] is utilized for obtaining the finite element solutions, which consumes more computational time while solving the resulting simultaneous equations.

References

[1] Pipes R.B., Pagano N.J.: Interlaminar stresses in composite laminates under uniform axial extension. Journal of Composite Materials, **4**, 538-548 (1970).
DOI: 10.1177/002199837000400409

[2] Bhaskar K., Varadhan T.K., Jacob C.: Free edge stresses in symmetric cross-ply plates undergoing flexure due to transverse loading. Journal of Aeronautical Society of India, **51**(3), 163-173 (1999).

[3] Wang A.S.D., Crossman F.W.: Some new results on edge effect in symmetric composite laminates. Journal of Composite Materials, **11**, 92-106 (1977).
DOI: 10.1177/002199837701100110

[4] Wang S.S., Choi L.: Boundary layer effects in composite laminates: Part-I Free edge stress singularities; Part-II Free edge stress solutions and basic characteristics. Transactions of ASME Journal of Applied Mechanics, **49**, 541-560 (1982).
DOI:10.1115/1.3162521

[5] Wei J., Zhao J.H.: Three-dimensional finite element analysis interlaminar stresses of symmetric laminates. Computers and Structures, **41** (4), 561-567 (1991).
DOI: 10.1016/0045-7949(91)90168-L

[6] Griffin, Jr O.H., Kamat M.P., Herakovich C.T.: Three-dimensional inelastic finite element analysis of laminated composites. Journal of Composite Materials, **5**, 543-560 (1981).
DOI:10.1177/002199838101500405

[7] Yamada Y., Okumua H.: Analysis of local stress in composite materials by the 3-D finite element in composite materials. Kawata J., Akasaka T., Eds., Proceedings of Japan-U.S. Conference, Tokyo (1981).

[8] Whitcomb J.D., Raju I.S., Goree J.G.: Reliability of the finite element method for calculating free edge stresses in composite laminates. Computers and Structures, **15**, 23-37 (1982).
DOI: 10.1016/0045-7949(82)90030-X

[9] Wang S.S., Stango R.J.: Optimally discretized finite elements for boundary layer stresses in composite laminates. American Institute of Aeronautics and Astronautics Journal, **21**, 614-620 (1983).

[10] Kim K.S., Hong C.S.: Delamination growth in angle-ply laminated composites. Journal of Composite Materials, **20**, 423-438 (1986).
DOI: 10.1177/002199838602000502

[11] Ye L.: Some characteristics of distributions of free edge interlaminar stresses in composite laminates. International Journal of Solids & Structures, **26**, 331- 351 (1990).
DOI:10.1016/0020-7683(90)90044-V

[12] Liu C.F., Jou H.S.: A new finite element formulations for stress analysis. Computers and Structures, **48**, 135-139 (1993).
DOI:10.1016/0045-7949(93)90464-O

[13] Chaudhuri R.A.: An equilibrium method for prediction of transverse shear stresses in a thick laminated plate. Computers and Structures, **23**, 139-146 (1986).

[14] Pagano N.J.: Exact solutions for composite laminates in cylindrical bending. Journal of Composite Materials, **3**, 398-411 (1969).
DOI: 10.1177/002199836900300304

[15] Klinkel S., Gruttmann F., Wagner W.: A continuum based 3D-shell element for laminated structures. Computers and Structures, **71**, 43-62 (1999).
Available at: http://www.solmech.tu-darmstadt.de/media/fachgebiet_festkoerpermechanik/veroeffentlichungen_prof_gruttmann/continuum_3d_se_1999.pdf

[16] Marimuthu R., Sundaresan M.K., Rao G.V.: Estimation of interlaminar stresses in laminated plates subjected to transverse loading using three-dimensional mixed finite element formulation. The Institution of Engineers (India), Technical Journal Aerospace Engineering, **84**, 1-8 (2003).

[17] Marimuthu R., Sundaresan M.K., Rao G.V.: Estimation of interlaminar normal and shear stresses by three-dimensional hexahedron element using mixed finite element formulation. The Institution of Engineers (India), Technical Journal Aerospace Engineering, **84**, 50-55 (2003).

[18] Auricchio F. Sacco E.: A mixed-enhanced finite-element for the analysis of laminated composite plates. International Journal for Numerical Methods in Engineering, **44**, 1481-1504 (1999).
DOI: 10.1002/(SICI)1097-0207(19990410)44:10<1481::AID-NME554>3.0.CO;2-Q

[19] Rohwer K.: Application of higher order theories to the bending analysis of layered composite plates. International Journal of Solids and Structures, **29**, 105-119 (1992).

[20] Reddy J.N.: A Generalization of Two-Dimensional Theories of Laminated Plates. Communications in Applied Numerical Methods, **3**, 173-180 (1987).
DOI: 10.1002/cnm.1630030303

[21] Carrera E.: Historical review of Zig-Zag theories for multilayered plates and shells. Applied Mechanics Reviews, **56**, 237-308 (2003)
DOI: 10.1115/1.1557614

[22] Reddy J.N.: On refined computational models of composite laminates. International Journal for Numerical Methods in Engineering, **27**, 361-382 (1989).
DOI: 10.1002/nme.1620270210

[23] Rao K.M., Meyer-Piening H.R.: Analysis of thick laminated anisotropic composite plates by the finite element method. Composite Structures, **15**, 185-213 (1990).

[24] Topdar P., Sheikh A.H., Dhang N.: Finite element analysis of composite and sandwich plates using a continuous interlaminar shear stress model. Journal of Sandwich Structures and Materials, **5**, 207-231 (2003).
DOI: 10.1177/1099636203005003001

[25] Gruttmann F., Wagner W.: Structural analysis of composite laminates using a mixed hybrid shell element. Computational Mechanics, **37**, 479-497 (2006).
DOI:10.1007/s00466-005-0730-1

[26] Schurg M., Wagner W., Gruttmann F.: An enhanced FSDT model for the calculation of interlaminar shear stresses in composite plate structures. Institute of Baustatik , University of Karlsruhe, Mitteilung **2** (2009).
Avalable at:
http://www.ibs.kit.edu/download/Mit_2009-02_Enhanced_FSDT_model interlaminar_shear_stresses_composite_plate_structures.pdf

[27] Hood P.: Frontal solution program for unsymmetric matrices. International Journal for Numerical Methods in Engineering, **10**, 379-399 (1976).
DOI: 10.1002/nme.1620100209